

## Section 11.6(a) Absolute Convergence and the Ratio and Root Tests.

given a Series  $\sum a_n$ , Consider the corresponding Series

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \dots \quad \text{whose terms are the}$$

absolute values of the original Series terms.

Definition A Series is called absolutely Convergent if the Series  $\sum |a_n|$  Converges.

Example ① Consider  $\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

We have  $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$  which is

Convergent (p-series,  $p=2 > 1$ ). So  $\sum \frac{(-1)^{n+1}}{n^2}$  is absolutely

Convergent.

Example ② Consider  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

we have  $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$  which is divergent

(p-series,  $p=1$ ) So  $\sum \frac{(-1)^{n+1}}{n}$  is not absolutely Convergent.

But it is Convergent! (previous Section).

SO, Convergence DOES NOT imply Absolute Convergence

Definition: A series  $\sum a_n$  is called "Conditionally Convergent" if it is Convergent, but NOT Absolutely Convergent.

So,  $\sum \frac{(-1)^{n+1}}{n}$  is Conditionally Convergent. (Ex 2)

• Here's a POWERFUL Theorem:

if a series is absolutely Convergent, Then it is Convergent!

Example 3 Test  $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$  for convergence or divergence.

The Series  $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^3} \right| = \sum \frac{|\sin(n)|}{n^3}$  satisfies

$\frac{|\sin(n)|}{n^3} \leq \frac{1}{n^3}$ ; by comparison test,  $\sum \frac{|\sin(n)|}{n^3}$  converges.

So,  $\sum \frac{\sin(n)}{n^3}$  is absolutely Convergent. By the

theorem above, it is Convergent.

## The ratio Test.

(i) if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$  Then  $\sum a_n$  is abs. Conv.  
and Therefore Convergent.

(ii) if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then  $\sum a_n$   
is divergent.

(iii) if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , The ratio test is inconclusive.

Example (4) Test  $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$  for Absolute Convergence.

Here  $a_n = (-1)^n \frac{n^3}{3^n}$ , So  $a_{n+1} = (-1)^{n+1} (n+1)^3 \cdot \frac{1}{3^{n+1}}$ .

$$\text{We have } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)^3 \cdot 3^n}{(-1)^n n^3 \cdot 3^{n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| (-1) \cdot \left( \frac{n+1}{n} \right)^3 \cdot \frac{1}{3} \right| = \frac{1}{3} < 1. \quad \text{So the}$$

Series Converges absolutely (and Thus Converges).

Example (5) Test  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$  for Convergence or divergence.

$$\text{Here } a_n = \frac{n^n}{n!} \quad \text{So } a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\text{So } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$\text{observe That } \frac{n!}{(n+1)!} = \frac{n \times (n-1) \times (n-2) \times \dots \times 2 \times 1}{(n+1) \times n \times (n-1) \times (n-2) \times \dots \times 2 \times 1} = \frac{1}{n+1}$$

$$\text{and } \frac{(n+1)^{n+1}}{n^n} = (n+1) \cdot \frac{(n+1)^n}{n^n} = (n+1) \cdot \left(\frac{n+1}{n}\right)^n.$$

$$\text{So, } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \cdot (n+1) \left(1 + \frac{1}{n}\right)^n \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

So the Series  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$  diverges.

More Examples of Ratio test:

$$\textcircled{1} \sum_{n=1}^{\infty} \frac{n^4}{6^n} \quad a_n = \frac{n^4}{6^n}, \quad a_{n+1} = \frac{(n+1)^4}{6^{n+1}} \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{6} \left(\frac{n+1}{n}\right)^4.$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{6} < 1 \Rightarrow \text{Series converges absolutely,}$$

and therefore converges.

$$\textcircled{2} \sum_{n=1}^{\infty} 5n! e^{-5n} \quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{5(n+1)!}{5n!} \cdot \frac{e^{-5(n+1)}}{e^{-5n}} \right| = \left| 5(n+1) e^{-5} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} 5(n+1) e^{-5} = \infty. \quad \text{Series diverges.}$$

## Section 11.6(b) (Continued)

### The root test

(i) if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ , Then The Series  $\sum a_n$  is absolutely Convergent, and therefore Convergent.

(ii) if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ , Then the Series  $\sum a_n$  is divergent.

(iii) if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , The root test is inconclusive.

Remark: if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , Then The Ratio test will also give a lim of 1 ( $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ ), and both tests are inconclusive (and vice-versa).

Examples ① Test the convergence of  $\sum_{n=1}^{\infty} \left( \frac{2n+3}{3n+2} \right)^n$ .

here  $a_n = \left( \frac{2n+3}{3n+2} \right)^n$ , so  $\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2}$ , and

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \frac{2}{3} < 1$$

Root.

Thus the Series Converges by the Ratio test.

② Test  $\sum_{n=1}^{\infty} \frac{n^n}{3^{2n+1}}$  for convergence or divergence.

$$a_n = \frac{n^n}{3^{2n+1}} = \frac{1}{3} \cdot \frac{n^n}{(3^2)^n} = \frac{1}{3} \cdot \left(\frac{n}{9}\right)^n; \quad \sqrt[n]{|a_n|} = \frac{1}{3^{1/n}} \cdot \frac{n}{9}$$

$\lim_{n \rightarrow \infty} \frac{1}{3^{1/n}} \cdot \frac{n}{9} = 1 \cdot \infty = \infty$  So Series diverges.

③  $\sum_{n=3}^{\infty} \frac{(-12)^n}{n}$ . Here it is easier to use the Ratio test.

$$a_n = \frac{(-12)^n}{n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(-12)^{n+1}}{n+1} \times \frac{n}{(-12)^n} = \frac{-12n}{n+1}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{12n}{n+1} \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 12 > 1$$

So The Series diverges.

④  $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$  Again The ratio test is the better choice.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-10)^{n+1}}{4^{2(n+1)+1}(n+1+1)} \times \frac{4^{2n+1}(n+1)}{(-10)^n} \right| = \left| \frac{-10 \times (n+1)}{4^2 (n+2)} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{-10}{4^2} \right| = \frac{10}{16} < 1 \Rightarrow \text{Series Converges}$$

⑤ does  $\sum_{n=1}^{\infty} \frac{(2n+1)!}{5^n (n!)^2}$  Converge or diverge? Ratio test.

$$a_n = \frac{(2n+1)!}{5^n (n!)^2} \quad a_{n+1} = \frac{(2n+3)!}{5^{n+1} [(n+1)!]^2} = \frac{(2n+3)(2n+2)(2n+1)!}{5^{n+1} (n+1)n! (n+1)n!}$$

$$\begin{aligned} \text{So, } \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(2n+3)(2n+2)(2n+1)!}{5^n \cdot 5 \cdot (n+1)^2 \cdot (n!)^2} \cdot \frac{5^n \cdot (n!)^2}{(2n+1)!} \right| \\ &= \left| \frac{(2n+3)(2n+2)}{5 \cdot (n+1)^2} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{4}{5} < 1 \end{aligned}$$

So the Series Converges.

⑥  $\sum_{n=1}^{\infty} \tan^{-1}(e^{-n})$ .  $a_n = \tan^{-1}(e^{-n})$ .  $\lim_{n \rightarrow \infty} a_n = \tan^{-1}(0) = 0$ .

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\tan^{-1}(e^{-(n+1)})}{\tan^{-1}(e^{-n})} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{0}{0} \text{ (L.H Rule)}$$

$$= \lim_{n \rightarrow \infty} \left| \frac{-e^{-n-1}}{1+e^{-2n-2}} \times \frac{1+e^{-2n}}{-e^{-n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^2}{e} \frac{1+e^{-2n}}{e^2+e^{-2n}} \right| = \frac{1}{e} < 1$$

So the Series Converges by the Ratio test.

⑦  $\sum_{k=1}^{\infty} \frac{\sqrt{3^k}}{2^k} = \sum_{n=1}^{\infty} \frac{(3^{\frac{1}{2}})^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{3}}{2}\right)^n$  Root test.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{\sqrt{3}}{2}\right)^n} = \frac{\sqrt{3}}{2} < 1. \text{ So the Series Converges.}$$

## More examples of Ratio test and Absolute Convergence.

$$\textcircled{1} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n}{3n^3+8} \quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{3(n+1)}{3(n+1)^3+8} \cdot \frac{3n^3+8}{3} \right|$$

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , So Ratio test is inconclusive. We must use a different test.

The Series  $\sum_{n=1}^{\infty} |a_n|$  satisfies  $\frac{3n}{3n^3+8} < \frac{3n}{3n^3} = \frac{1}{n^2}$  for all

$n \geq 1$ . by the Comparison test,  $\sum_{n=1}^{\infty} |a_n|$  Converges,

which implies that  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n}{3n^3+8}$  Converges absolutely.

$$\textcircled{2} \quad \sum_{n=1}^{\infty} \frac{(-60)^n}{n!} \quad \lim_{n \rightarrow \infty} \left| \frac{(-60)^{n+1} n!}{(n+1)! (-60)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{60}{n+1} \right| = 0 < 1$$

By the Ratio test, the Series Converges absolutely.

$$\textcircled{3} \quad \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{8n}. \quad \text{Observe that } \cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

This means that  $\cos(n\pi) = (-1)^n$ .

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{8n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{8n}. \quad \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{8(n+1)} \cdot \frac{8n}{(-1)^n} \right| = 1.$$

Ratio test is Inconclusive. Series diverges or Converges (so far)

Let's look at  $\sum_{n=1}^{\infty} \frac{(-1)^n}{8^n}$ . it is an alternating series with

$$b_n = \frac{1}{8^n}, \quad \text{so } b_{n+1} < b_n, \quad \text{and } \lim_{n \rightarrow \infty} b_n = 0.$$

So the series converges (not absolutely) by the alternating series test.

$$\textcircled{4} \quad \sum_{n=1}^{\infty} (-1)^n \frac{(7n)!}{7^n n! n} \quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} (7(n+1))!}{7^{n+1} (n+1)! (n+1)} \cdot \frac{7^n n! n}{(-1)^n (7n)!} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty. \quad \text{Series diverges by Ratio test.}$$

$$\textcircled{5} \quad \sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 2^n}{(2n+1)!} \quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!^2 2^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(n!)^2 2^n} \right|$$

$$= \left| \frac{(n+1)^2 \cdot 2}{(2n+3)(2n+2)} \right| \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{4} = \frac{1}{2} < 1$$

Series converges absolutely by the Ratio Test.